International Journal of Theoretical Physics, Vol. 25, No. 5, 1986

Gauge Group of Gravity, Spinors, and Anomalies

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Received October 15, 1985

A discussion is given of the gravitational anomalies that arise from coupling Weyl spinors to gravity, treating the metric, the soldering form, and the connection as independent dynamical variables. This system is strictly analogous to Weyl spinors coupled to Yang–Mills fields and a nonlinear sigma model. The larger gauge group of this formulation is seen to lie at the root of the equivalence between Einstein and Lorentz anomalies.

1. INTRODUCTION

General relativity is usually studied in the metric formulation, in which the fundamental dynamical variable is the Riemannian structure $g_{\mu\nu}$. For some purposes, (in particular to define spinors) it is necessary to go over to the n-bein formulation, in which the fundamental dynamical variable is the soldering form $\bar{\theta}^{\bar{m}}{}_{\mu}$. Here, $\bar{\theta}$ is regarded as an isomorphism from the tangent bundle TM, with coordinate bases $\{\partial_\mu\}$, to an "internal" vector bundle $\bar{\xi}$ with fibers $\mathscr{R}^n$, $n = \dim M$, and bases $\{\bar{e}_{\bar{m}}\}$ orthonormal with respect to a fixed fiber metric. The soldering form $\bar{\theta}$ can be used to pull back the fiber metric in $\bar{\xi}$ to TM:

$$g_{\mu\nu} = \bar{\theta}^{\bar{m}}{}_{\mu}\bar{\theta}^{\bar{n}}{}_{\nu}\delta_{\bar{m}\bar{n}} \tag{1}$$

Conversely, the Riemannian structure g determines the soldering form $\bar{\theta}$ up to a local $O(n)$ gauge transformation. Therefore, when we go from the metric to the n-bein formulation, in order not to introduce new physical degrees of freedom, we have to enlarge the original gauge group Diff M to the group $\mathrm{Aut}^{O(n)}\bar{\xi}$ of orthogonal automorphisms of $\bar{\xi}$ (covering arbitrary diffeomorphisms of M). For reasons that have been discussed elsewhere (Percacci, 1982, 1984), it is convenient to go one step further and to treat

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the soldering form and the metric as independent dynamical variables and at the same time enlarge the gauge group so that this new formulation is equivalent to the old. This has also been discussed by Komar (1985) based on completely different motivations. For notational reasons, let us call ξ the vector bundle that was previously called $\bar{\xi}$ and let $\{e_n\}$ be local bases in ξ; we reserve the barred indices for orthonormal frames, to be introduced below. In the new formulation, the fiber metric in ξ is no longer fixed, but is a dynamical variable κ_{mn} (a set of scalars under coordinate transformations), along with the soldering form $\theta^m{}_\mu$. We can again pull back the fiber metric of ξ by means of θ to get a Riemannian structure on M:

$$g_{\mu\nu} = \theta^m{}_\mu \theta^n{}_\nu \kappa_{mn} \tag{2}$$

Notice that now it is no longer possible to identify the soldering form with the n-bein of g. In this formulation, the gauge group is $\mathrm{Aut}^{GL(n)}\xi$. We can fix the $GL(n)$ gauge in such a way that either $\theta^m{}_\mu = \delta^m{}_\nu$, thus recovering the original metric formulation, or $\kappa_{mn} = \delta_{mn}$, thus recovering the n-bein formulation (Percacci, 1982, 1984).

This generalized formulation is again unsuitable for the treatment of spinors, because spinors are only defined for the orthogonal groups. Therefore, if we want to couple spinors to gravity and at the same time preserve the full $\mathrm{Aut}^{GL(n)}\xi$ gauge group, it is necessary to define n-beins for the fiber metric κ. Just as we regarded the n-bein for g as an isomorphism of TM to $\bar{\xi}$, so we will regard the n-bein for κ as an isomorphism τ of ξ to another vector bundle $\bar{\xi}$ with a fixed fiber metric and orthonormal bases $\{\bar{e}_{\bar{m}}\}$. Then κ is the pullback of this fixed fiber metric by τ:

$$\kappa_{mn} = \tau^{\bar{r}}{}_m \tau^{\bar{s}}{}_n \delta_{\bar{r}\bar{s}} \tag{3}$$

If we define the vector-bundle isomorphism $\bar{\theta}$ from TM to $\bar{\xi}$ by composing τ with θ:

$$\bar{\theta}^{\bar{m}}{}_\rho = \tau^{\bar{m}}{}_r \theta^r{}_\rho \tag{4}$$

then, from equations (2) and (3), equation (1) follows again. The gauge group will now consist, roughly speaking, of $O(n)$ Yang–Mills type transformations acting on the barred Latin indices, $GL(n)$ Yang–Mills type transformations acting on the unbarred Latin indices, and diffeomorphisms acting on the Greek indices. I describe this gauge group more precisely in Section 2. In Section 3 I discuss the anomalies for a system consisting of a G Yang–Mills field coupled to a G/H nonlinear sigma model. This system is very closely related to theories of gravity and it will be seen that the properties of its anomalies are shared by gravitational anomalies.

In Section 4 I first review the classical Dirac equation and its currents in a background gravitational field with torsion. It will be seen that the gravitational coupling of spinors is essentially the same as their coupling to a gauged nonlinear sigma model. I shall denote by ∇ a covariant derivative in ξ such that

$$\nabla_\mu \kappa_{mn} = 0 \tag{5}$$

The components of ∇ will be denoted $\omega_\lambda{}^m{}_n$. The derivative ∇ can be pulled back by means of θ to a covariant derivative D in TM whose components are

$$\Gamma_\lambda{}^\mu{}_\nu = \theta_m{}^\mu \omega_\lambda{}^m{}_n \theta^n{}_\nu + \theta_m{}^\mu \partial_\lambda \theta^m{}_\nu \tag{6}$$

Clearly, D is also metric, in the sense that $D_\lambda g_{\mu\nu} = 0$. Finally, ∇ can be regarded as the pullback by τ of a covariant derivative $\bar{\nabla}$ in $\bar{\xi}$ with components

$$\bar{\omega}_\lambda{}^{\bar{m}}{}_{\bar{n}} = \tau^{\bar{m}}{}_r \omega_\lambda{}^r{}_s \tau_{\bar{n}}{}^s + \tau^{\bar{m}}{}_r \partial_\lambda \tau_{\bar{n}}{}^r \tag{7}$$

The metricity condition is equivalent to the statement that $\bar{\omega}_{\lambda\bar{m}\bar{n}} = -\bar{\omega}_{\lambda\bar{n}\bar{m}}$. The torsion of ∇ is the covariant exterior derivative of θ:

$$\Theta_\mu{}^m{}_\nu = \partial_\mu \theta^m{}_\nu - \partial_\nu \theta^m{}_\mu + \omega_\mu{}^m{}_n \theta^n{}_\nu - \omega_\nu{}^m{}_n \theta^n{}_\mu \tag{8}$$

The torsion of $\bar{\nabla}$ is given by the same formula with $\bar{\theta}$ replacing θ and $\bar{\omega}$ replacing ω; the torsion of D is given by $\Theta_\mu{}^\rho{}_\nu = \Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu}$. I then discuss the geometrical and physical meaning of gravitational anomalies (Alvarez-Gaumé and Witten, 1983) in this generalized setting. The inclusion of torsion is quite straightforward and does not change the physical meaning of anomalies. The larger gauge group is seen to lead, via different partial gauge fixings, to the so-called equivalence of diffeomorphism and local $O(n)$ anomalies (Bardeen and Zumino 1984).

2. THE GAUGE GROUP AND ITS LIE ALGEBRA

Let us denote Iso(TM, ξ) the space of all isomorphisms from TM to ξ (soldering forms θ), Iso$(\xi, \bar{\xi})$ the space of all isomorphisms from ξ to $\bar{\xi}$ (n-beins τ), Riem ξ the space of fiber metrics κ in ξ, and $\mathscr{C}(\xi)$ the space of linear connections ω in ξ. Since we are going to interpret the gauge transformations from an active point of view, the bundles TM, ξ and $\bar{\xi}$ come with fixed local bases $\{\partial_\mu\}$, $\{e_m\}$, and $\{\bar{e}_{\bar{m}}\}$. When we go over from the formulation based on the variables ω, θ, κ to the one based on the variables ω, θ, τ, we have to enlarge the gauge group to include $\mathrm{Aut}_M^{O(n)} \bar{\xi}$, the vertical orthogonal automorphisms of $\bar{\xi}$. The new gauge group is the group

$$\mathrm{Aut}^{GL(n)\times O(n)} \xi \oplus \bar{\xi}$$

(denoted Aut $\xi\oplus\bar{\xi}$ for notational simplicity). It consists of triples $(u, \bar{u}, f)$ with $f\in \mathrm{Diff}\ M$, $(u,f)\in \mathrm{Aut}^{GL(n)}\xi$, and $(\bar{u},f)\in \mathrm{Aut}^{O(n)}\bar{\xi}$. The group $\mathrm{Aut}_M^{GL(n)}\xi$ is a normal subgroup with the obvious injection $\alpha:(u, Id_M)\mapsto(u, Id_{\bar{\xi}}, Id_M)$. There is a short exact sequence:

$$1\to \mathrm{Aut}_M^{GL(n)}\xi \xrightarrow{\alpha} \mathrm{Aut}\ \xi\oplus\bar{\xi} \xrightarrow{\beta} \mathrm{Aut}^{O(n)}\bar{\xi}\to 1 \tag{9}$$

where $\beta:(u, \bar{u}, f)\mapsto(\bar{u}, f)$. For each $\theta\in \mathrm{Iso}(TM, \xi)$ we can define an injection $\gamma^\theta: \mathrm{Aut}^{O(n)}\bar{\xi}\to \mathrm{Aut}\ \xi\oplus\bar{\xi}$ by

$$\gamma^\theta:(\bar{u},f)\mapsto(\theta\circ Tf\circ\theta^{-1}, \bar{u}, f)$$

For each $\tau\in \mathrm{Iso}(\xi, \bar{\xi})$ we can define an injection γ^τ by

$$\gamma^\tau:(\bar{u},f)\to(\tau^{-1}\circ\bar{u}\circ\tau, \bar{u}, f)$$

All these homomorphisms split the sequence (9) and hence define semidirect product structures in Aut $\xi\oplus\bar{\xi}$. Also note the injection $\gamma: \mathrm{Aut}_M^{O(n)}\bar{\xi}\to \mathrm{Aut}\ \xi\oplus\bar{\xi}$ given by $\gamma(\bar{u}, Id_M)=(Id_\xi, \bar{u}, Id_M)$.

In general, it is not possible to define a "natural" subgroup $\mathrm{Aut}^{GL(n)}\xi$. One needs additional structures. For instance, if M is parallelizable, and a global trivialization of $\bar{\xi}$ is given, we can embed $\mathrm{Aut}^{GL(n)}\xi$ in Aut $\xi\oplus\bar{\xi}$ by $\tilde{\alpha}:(u,f)\mapsto(u,\tilde{f},f)$, where $\tilde{f}$ denotes the (trivialization dependent) lift of f: $\tilde{f}(x, a)=(f(x), a)$.

The gauge group acts on the fields from the right as follows:

$$\theta\mapsto\theta'=u^{-1}\circ\theta\circ Tf \tag{10}$$

$$\tau\mapsto\tau'=\bar{u}^{-1}\circ\tau\circ u \tag{11}$$

$$\nabla\mapsto\nabla', \qquad \nabla'_v\sigma=u^{-1}(\nabla_{Tf(v)}(u\circ\sigma\circ f^{-1})) \tag{12}$$

Let $\Lambda: U\to GL(n)$ be the local representative of (u,f) on the open set U, defined by

$$u(e_m(f^{-1}(x)))=e_n(x)\Lambda^n{}_m(x)$$

and similarly $\bar{\Lambda}: U\to O(n)$ be the local representative of $(\bar{u},f)$. Furthermore, let $x'=f^{-1}(x)$. The transformation laws become

$$\theta'^m{}_\mu(x')=\Lambda^{-1m}{}_n(x)\theta^n{}_\nu(x)\frac{\partial x^\nu}{\partial x'^\mu} \tag{13}$$

$$\tau'^{\bar{m}}{}_n(x')=\bar{\Lambda}^{-1m}{}_{\bar{r}}(x)\tau^{\bar{r}}{}_s(x)\Lambda^s{}_n(x) \tag{14}$$

$$\omega'_\lambda{}^m{}_n(x')=\frac{\partial x^\tau}{\partial x'^\lambda}\left(\Lambda^{-1m}{}_r(x)\omega_\tau{}^r{}_s(x)\Lambda^s{}_n(x)+\Lambda^{-1m}{}_r(x)\frac{\partial}{\partial x^\tau}\Lambda^r{}_n(x)\right) \tag{15}$$

If we assume appropriate boundary conditions on the fields and the gauge transformations, this action is free and we have a principal bundle with structure group $\mathscr{G} = \text{Aut}\,\xi \oplus \bar{\xi}$, total space

$$\mathscr{E} = \mathscr{C}(\xi) \times \text{Iso}(TM, \xi) \times \text{Iso}(\xi, \bar{\xi})$$

and base space $\mathscr{E}/\mathscr{G}$. The configuration space of the gravitational field is the subspace $\mathscr{Q}$ of $\mathscr{E}/\mathscr{G}$ defined by the constraint (5).

One sees immediately from the transformation properties that the subgroup $\text{Aut}_M^{GL(n)}\xi$ acts transitively and freely on $\text{Iso}(TM, \xi)$; the stabilizer of a given θ under the action of $\mathscr{G}$ is the subgroup $\gamma^\theta\ \text{Aut}^{O(n)}\bar{\xi}$. Similarly, $\text{Aut}_M^{GL(n)}\xi$ acts transitively and freely on $\text{Iso}(\xi, \bar{\xi})$, and the stabilizer of a given τ under the action of $\mathscr{G}$ is the subgroup $\gamma^\tau\ \text{Aut}^{O(n)}\bar{\xi}$. Therefore, we can also write

$$\mathscr{Q} = (\mathscr{C}(\bar{\xi}) \times \text{Iso}(TM, \bar{\xi}))/\text{Aut}^{O(n)}\bar{\xi}$$

where $\mathscr{C}(\bar{\xi})$ denotes the space of $O(n)$ connections in $\bar{\xi}$. Let P and $\bar{P}$ be the (total spaces of) the principal $GL(n)$ and $O(n)$ bundles associated to ξ and $\bar{\xi}$. The Lie algebra of (left-invariant vector fields on) $\text{Aut}\,\xi \oplus \bar{\xi}$ is isomorphic as a vector space to $T_{Id}(\text{Aut}\,\xi \oplus \bar{\xi})$, which is the space of $GL(n) \times O(n)$-invariant vectorfields on the fiber product

$$P \underset{M}{\times} \bar{P}$$

[i.e., triples $(v, \bar{v}, w)$ with w a vector field on M, v a $GL(n)$-invariant vector field on P projecting on w, $\bar{v}$ a $O(n)$-invariant vector field on $\bar{P}$ projecting on w]. The space of $GL(n) \times O(n)$-invariant vector fields on

$$P \underset{M}{\times} \bar{P}$$

is itself an algebra with bracket

$$\begin{aligned}[(v_1, \bar{v}_1, w_1), (v_2, \bar{v}_2, w_2)]& \\ = ([v_1, v_2], [\bar{v}_1, \bar{v}_2], [w_1, w_2])& \qquad (16)\end{aligned}$$

Let $\rho(v, \bar{v}, w)$ and $\lambda(v, \bar{v}, w)$ be the right- and left-invariant vector fields on $\text{Aut}\,\xi \oplus \bar{\xi}$ which coincide with $(v, \bar{v}, w)$ at the identity. It can be shown that ρ is an isomorphism of algebras, so λ is an antiisomorphism:

$$\begin{aligned}[\lambda(v_1, \bar{v}_1, w_1), \lambda(v_2, \bar{v}_2, w_2)]& \\ = -\lambda([(v_1, \bar{v}_1, w_1), (v_2, \bar{v}_2, w_2)])& \qquad (17)\end{aligned}$$

Given the local bases in ξ and $\bar{\xi}$, we have local trivializations in P and $\bar{P}$. We can accordingly split v and $\bar{v}$ into parts tangent to the fibers and to M.

Let us represent $(v, \bar{v}, w)$ as $(\varepsilon, \bar{\varepsilon}, w)$, where $\varepsilon : M \to \mathrm{Lie}GL(n)$ and $\bar{\varepsilon} : M \to \mathrm{Lie}O(n)$ are such that at each point $p \sim (x, a) \in M \times GL(n)$, $v(p) = w(x) +$ (the value at a of the right $GL(n)$-invariant vector field that coincides with $\varepsilon(x)$ at the origin), and similarly with $\bar{v}$. Since w is independent of the coordinate in the fiber, we have

$$\begin{aligned}
&[(\varepsilon_1, \bar{\varepsilon}_1, w_1), (\varepsilon_2, \bar{\varepsilon}_2, w_2)] \\
&\quad = (-[\varepsilon_1, \varepsilon_2] + w_1{}^\mu \partial_\mu \varepsilon_2 - w_2{}^\mu \partial_\mu \varepsilon_1, \\
&\qquad -[\bar{\varepsilon}_1, \bar{\varepsilon}_2] + w_1{}^\mu \partial_\mu \bar{\varepsilon}_2 - w_2{}^\mu \partial_\mu \bar{\varepsilon}_1, [w_1, w_2])
\end{aligned} \tag{18}$$

(The minus signs arise because ε and $\bar{\varepsilon}$ are left-invariant, while v and $\bar{v}$ are right-invariant.) Putting together these formulas, we obtain the brackets in the Lie algebra of Aut $\xi \oplus \bar{\xi}$ in terms of the local components $(\varepsilon, \bar{\varepsilon}, w)$:

$$\begin{aligned}
&[\lambda(\varepsilon_1, \bar{\varepsilon}_1, w_1), \lambda(\varepsilon_2, \bar{\varepsilon}_2, w_2)] \\
&\quad = \lambda([\varepsilon_1, \varepsilon_2] - w_1{}^\mu \partial_\mu \varepsilon_2 + w_2{}^\mu \partial_\mu \varepsilon_1, [\bar{\varepsilon}_1, \bar{\varepsilon}_2] - w_1{}^\mu \partial_\mu \bar{\varepsilon}_2 \\
&\qquad + w_2{}^\mu \partial_\mu \bar{\varepsilon}_1, -[w_1, w_2])
\end{aligned} \tag{19}$$

We notice that the vertical vector fields $(\varepsilon, \bar{\varepsilon}, 0)$ form an ideal corresponding to the normal subgroup

$$\mathrm{Aut}_M \xi \oplus \bar{\xi} = \mathrm{Aut}_M^{GL(n)} \xi \times \mathrm{Aut}_M^{O(n)} \bar{\xi}.$$

If M is parallelizable and fixed global trivializations of ξ and $\bar{\xi}$ are given, it becomes meaningful to talk about the vector fields of the form $(0, 0, w)$ and they are seen to form a subalgebra [in general a vector field of the form $(0, 0, w)$ in one trivialization would have a vertical component in another trivialization]. If Diff M is regarded as a subgroup of $\mathrm{Aut}^{GL(n)} \xi$, this subalgebra corresponds to the subgroup $\tilde{\alpha}$ Diff M. In the absence of a global trivialization or other additional structures, the gauge algebra does not contain any "natural" subalgebra isomorphic to the algebra of the diffeomorphism group.

The Lie algebra of Aut $\xi \oplus \bar{\xi}$ is realized as vector fields on the space $\mathscr{E}$. We denote by $\delta(\varepsilon, \bar{\varepsilon}, w)$ the fundamental vector field generated by $\lambda(\varepsilon, \bar{\varepsilon}, w)$. These vector fields again obey the algebra (19). This can also be verified more directly as follows. Under an infinitesimal gauge transformation $x'^\mu = x^\mu - w^\mu$, $\Lambda^m{}_n = \delta^m{}_n + \varepsilon^m{}_n$, $\bar{\Lambda}^{\bar{m}}{}_{\bar{n}} = \delta^{\bar{m}}{}_{\bar{n}} + \bar{\varepsilon}^{\bar{m}}{}_{\bar{n}}$ the fields transform by

$$\delta\theta^m{}_\mu = w^\lambda \partial_\lambda \theta^m{}_\mu + \theta^m{}_\nu \partial_\mu w^\nu - \varepsilon^m{}_n \theta^n{}_\mu \tag{20}$$

$$\delta\tau^{\bar{m}}{}_n = w^\lambda \partial_\lambda \tau^{\bar{m}}{}_n - \bar{\varepsilon}^{\bar{m}}{}_{\bar{r}} \tau^{\bar{r}}{}_n + \tau^{\bar{m}}{}_r \varepsilon^r{}_n \tag{21}$$

$$\delta\omega_\tau{}^m{}_n = w^\lambda \partial_\lambda \omega_\tau{}^m{}_n + \omega_\lambda{}^m{}_n \partial_\tau w^\lambda + \nabla_\tau \varepsilon^m{}_n \tag{22}$$

Decomposing

$$\delta(\varepsilon, \bar{\varepsilon}, w) = \delta\theta \frac{\delta}{\delta\theta} + \delta\tau \frac{\delta}{\delta\tau} + \delta\omega \frac{\delta}{\delta\omega}$$

we can explicitly compute the functional brackets and show that they obey the algebra (19).

3. AN ANALOGY

Consider the coupling of Weyl fermions to a background YM field A on a G vector bundle ξ. The Dirac operator is a map $\not{D}_A:\Gamma(\eta_+\otimes\xi)\to\Gamma(\eta_-\otimes\xi)$, where $\eta_\pm$ denote the positive and negative chirality spinor bundles. There are inherent ambiguities in the definition of det $\not{D}_A$ and therefore the one-loop effective action $\Gamma(A)=-\ln\det\not{D}_A$, when regarded as a function on $\mathscr{C}(\xi)$, is not constant on the orbits of the gauge group $\mathscr{G}=\mathrm{Aut}^G_M\xi$. The anomaly is the differential of Γ along the orbits. Det $\not{D}_A$ is a smooth, complex function with constant modulus on each orbit (Alvarez-Gaumé and Witten, 1983); if $\mathscr{G}$ is multiply connected, the phase of det $\not{D}_A$ may vary by a multiple of $2\pi i$ as one goes through a noncontractible loop, and hence Γ may not be a single-valued function on the orbit. Therefore the anomaly should be thought of as a closed but not necessarily exact one-form α^G on the gauge group: $\alpha^G(\varepsilon)=\delta_\varepsilon\Gamma=\int\varepsilon^i\nabla_\nu\langle J^\nu_i\rangle$, where $\langle J^\nu_i\rangle=\delta\Gamma/\delta A^i_\nu$. The condition that $d\alpha=0$ is known in the physical literature as the Wess-Zumino (1971) consistency condition. The exact coefficient of the anomaly can only be determined by an explicit calculation, but its general form can be found by solving the Wess-Zumino consistency condition in terms of secondary characteristic classes (Atiyah and Singer, 1984; Alvarez-Gaumé and Ginsparg, 1986).

As a result of the Wess-Zumino consistency condition, the effective action is known everywhere on the orbit mod $2\pi i$ once it is known at a point:

$$\Gamma(A^g)=\Gamma(A)+\int_1^g\alpha^G \tag{23}$$

where A^g denotes the gauge transform of A and the integral is along any path in $\mathscr{G}$ joining the identity to g.

What happens when the system is further coupled to a nonlinear sigma model? Consider first the case when the NSM has values in G. One has to assume that ξ is trivial for the nonlinear scalar field to be globally defined. The bosonic configuration space is then $(\mathscr{C}(\xi)\times\Gamma(\xi_G))/\mathrm{Aut}^G_M\xi$; here ξ_G denotes the principal G-bundle associated to ξ. Since the gauge group acts freely and transitively on $\Gamma(\xi_G)$, this is just $\mathscr{C}(\xi)$ and the gauge group is reduced to the identity. This kinematical fact suggests that there should be no anomaly for this system. Indeed, let us define the Weyl operator

$$\hat{\not{D}}_{(A,g)}:\Gamma(\eta_+\otimes\xi)\to\Gamma(\eta_-\otimes\xi)$$

for fixed $A\in\mathscr{C}(\xi)$, $g\in\Gamma(\xi_G)$ by

$$\hat{\not{D}}_{(A,g)}=g^{-1}\not{D}_A g=\not{D}_{A^g} \tag{24}$$

Under a gauge transformation $\bar{g}$, $g \mapsto \bar{g}^{-1}g$, so this operator is gauge invariant:

$$\hat{\not{D}}_{(A^{\bar{g}}, g^{\bar{g}})} = \not{D}_{(A^{\bar{g}})^{\bar{g}^{-1}g}} = \not{D}_{A^g} = \hat{\not{D}}_{(A,g)} \tag{25}$$

As a consequence, $\hat{\Gamma}(A, g) = -\ln \det \hat{\not{D}}_{(A,g)}$ will be constant on the orbits of $\mathcal{G}$, i.e., there will be no anomaly. From equation (24), $\hat{\Gamma}(A, g) = \Gamma(A^g)$, so from equation (23)

$$\hat{\Gamma}(A, g) = \Gamma(A) + \int_1^g \alpha^G \tag{26}$$

The second term on the rhs is the Wess–Zumino–Witten action (Wess and Zumino, 1971; Witten, 1983; Alvarez-Gaumé and Ginsparg, 1986). This formula is usually interpreted in the following way: suppose the spinors were not directly coupled to the NSM, i.e., $\hat{\not{D}}_{(A,g)} = \not{D}_A$. Then, the theory would have the same anomalies as the pure gauge case. However, in the presence of the NSM there exists a local functional of A and g that can be added as a counterterm to the effective action and whose gauge variation exactly cancels the anomaly. The interpretation presented above will be seen to be more convenient for the sake of comparison with gravity.

The discussion in the preceding section motivates us to consider the more complicated case when G is a semidirect product $G = K \ltimes H$ (i.e., there is a split exact sequence $1 \to K \to G \rightleftarrows H \to 1$). We couple the G Yang–Mills field to an NSM with values in G/H. The elements of G are couples $g = (k, h)$ and the multiplication is

$$(k, h)(k', h') = (k \cdot \psi_h(k'), hh')$$

where $\psi : H \to \mathrm{Aut}\, K$ is the homomorphism defining the semidirect product structure. The map $G/H \to K$ given by $(k, h)H \mapsto k$ is a diffeomorphism and the action of K as a subgroup of G on G/H is the same as left multiplication in K. So, given $\phi = gH$, there is a unique $k \in K$ such that $\phi = k(eH)$.

Let P be the (total space of the) principal G bundle associated to ξ. Then $R = P/H$ is the (total space of the) associated bundle $\xi_{G/H}$; $Q = P/K$ is naturally the total space of a principal H bundle ζ with action $(pK)h = (ph)K$. The configurations of the NSM are sections of $\xi_{G/H}$; since $\xi_{G/H}$ is isomorphic to a principal K bundle, this means that it has to be globally trivializable. If a G-automorphism of P is of the form $(k, 1)$ in one trivialization, then it is of the same form in any trivialization. These automorphisms form a subgroup denoted $\mathrm{Aut}_M^K \xi$. There is an exact sequence

$$1 \to \mathrm{Aut}_M^K \xi \to \mathrm{Aut}_M^G \xi \to \mathrm{Aut}_M^H \zeta \to 1 \tag{27}$$

Each section ϕ of $\xi_{G/H}$ defines an embedding of Q into P and hence a homomorphism $\gamma^{\phi}: \mathrm{Aut}_M^H \zeta \to \mathrm{Aut}_M^G \xi$ which splits the sequence (27). If ϕ_0 is defined by $\phi_0(x) = eH$ (in all charts), the automorphisms in the image of γ^{ϕ_0} are those whose local representatives are of the form $(1, h)$ with $h: M \supset U \to H$.

The configuration space of the system is

$$\mathcal{Q} = (\mathcal{C}(\xi) \times \Gamma(\xi_{G/H}))/\mathrm{Aut}_M^G \xi$$

We can choose a "unitary" gauge such that $\phi = \phi_0$; this leaves a residual gauge freedom $\gamma^{\phi_0}\,\mathrm{Aut}_M^H \zeta$, so

$$\mathcal{Q} = \mathcal{C}(\xi)/\gamma^{\phi_0}\,\mathrm{Aut}_M^H \zeta$$

Again, this kinematical argument suggests that in the unitary gauge defined by ϕ_0 any anomaly should be in the image of γ^{ϕ_0}. Given any $\phi \in \Gamma(\xi_{G/H})$, let k be the unique element of $\mathrm{Aut}_M^K \xi$ such that $k^{-1}\phi = \phi_0$. The Weyl operator

$$\hat{\not{D}}_{(A,\phi)}: \Gamma(\eta_+ \otimes \xi) \to \Gamma(\eta_- \otimes \xi)$$

is defined by

$$\hat{\not{D}}_{(A,\phi)} = k^{-1} \not{D}_A k = \not{D}_{A^k} \tag{28}$$

Under a gauge transformation $\bar{k} \in \mathrm{Aut}_M^K \xi$, $\phi^{\bar{k}} = \bar{k}^{-1}\phi$, so

$$\hat{\not{D}}_{(A^{\bar{k}}, \phi^{\bar{k}})} = \not{D}_{(A^{\bar{k}})^{\bar{k}^{-1}k}} = D_{A^k} = \hat{\not{D}}_{(A,\phi)} \tag{29}$$

Under a transformation $\bar{h}$ belonging to the image of γ^{ϕ_0}, we have

$$\phi^{\bar{h}} = \bar{h}^{-1}\phi = (1, \bar{h}^{-1})(k, h)H = (\psi_{\bar{h}^{-1}}(k), \bar{h}^{-1}h)H$$

and furthermore

$$(1, \bar{h})(\psi_{\bar{h}^{-1}}(k), 1) = (k, 1)(1, \bar{h})$$

so

$$\hat{\not{D}}_{(A^{\bar{h}}, \phi^{\bar{h}})} = \hat{\not{D}}_{(A^{\bar{h}})^{\psi_{h^{-1}}(k)}} = \not{D}_{(A^k)^{\bar{h}}} = \bar{h}^{-1} \hat{\not{D}}_{(A,\phi)} \bar{h} \tag{30}$$

So the Weyl operator is invariant under $\mathrm{Aut}_M^K \xi$ and covariant under gauge transformations in the stabilizer of ϕ_0. Consequently, the effective action will be constant along the orbits of $\mathrm{Aut}_M^K \xi$ and, as expected, any anomaly will be in the subgroup $\gamma^{\phi_0}\,\mathrm{Aut}_M^H \zeta$. Since $\hat{\Gamma}(A, \phi) = \Gamma(A^k)$, we have

$$\hat{\Gamma}(A, \phi) = \Gamma(A) + \int_1^k \alpha^K \tag{31}$$

where α^K denotes the anomaly for K. As in the case $H = \{e\}$, we could interpret this formula by saying that in the presence of a K-valued NSM the component of the gauge anomaly in the subgroup $\mathrm{Aut}_M^K \xi$ can be canceled by adding a local functional of A and ϕ.

If we change the unitary gauge by choosing a different local trivialization so that the standard section is ϕ_0', the anomaly is transferred to the subgroup $\gamma_{\phi_0'}\,\mathrm{Aut}\,{}^H_M\zeta$. In equation (31) this is effected by adding the term $\int_k^{k'}\alpha^K$, where k' is the gauge transformation such that $\phi^{k'}=k'^{-1}\phi=\phi_0'$.

To summarize the result of this discussion: I call a Goldstone boson (in a generalized sense) a field whose configuration space is a homogeneous space for the gauge group. I call a unitary gauge (in a generalized sense) a gauge in which the Goldstone boson is constant. In a unitary gauge, the anomaly lies in the stability subgroup; different unitary gauges are related by gauge transformations in the anomaly-free normal subgroup. The anomaly is moved from one stability subgroup to another by adding suitable Wess–Zumino counterterms. We shall see that exactly the same phenomena occur in gravity.

4. GRAVITATIONAL ANOMALIES

Assume that M has even dimension n and admits a unique spin structure (this is the case, e.g., if M is simply connected). The spin structure is a principal Spin(n)-bundle with total space Q and a bundle homomorphism $Q\to\bar{P}$ which is a double cover. I call η_+ and η_- the associated bundles of positive- and negative-chirality Weyl spinors; $\eta=\eta_+\oplus\eta_-$ is the bundle of Dirac spinors. Relative to a local field of orthonormal frames $\{\bar{e}_{\bar{m}}\}$ in $\bar{\xi}$, one has Dirac matrices $\bar{\gamma}^{\bar{m}}$ satisfying $\{\bar{\gamma}^{\bar{m}},\bar{\gamma}^{\bar{n}}\}=2\delta^{\bar{m}\bar{n}}$; the generators of Spin$(n)$ in the Dirac representation are $\bar{\sigma}^{\bar{m}\bar{n}}=\frac{1}{4}[\bar{\gamma}^{\bar{m}},\bar{\gamma}^{\bar{n}}]$. The connection $\bar{\nabla}$ in $\bar{\xi}$ lifts to a "spin connection" in η denoted by the same symbol and with the same local components $\bar{\omega}_\lambda{}^{\bar{m}}{}_{\bar{n}}$. The covariant derivatives of spinors and conjugate spinors are

$$\bar{\nabla}_\lambda\psi=\partial_\lambda\psi+\tfrac{1}{2}\bar{\omega}_{\lambda\bar{m}\bar{n}}\sigma^{\bar{m}\bar{n}}\psi \tag{32}$$

$$\bar{\nabla}_\lambda\bar{\psi}=\partial_\lambda\bar{\psi}-\tfrac{1}{2}\bar{\omega}_{\lambda\bar{m}\bar{n}}\bar{\psi}\bar{\sigma}^{\bar{m}\bar{n}} \tag{33}$$

The action for ψ and $\bar{\psi}$, treated as independent variables, is

$$S(\psi,\bar{\psi};\bar{\omega},\bar{\theta})=\int d^4x\,\sqrt{g}\,\{\tfrac{1}{2}i(\bar{\psi}\bar{\gamma}^{\bar{m}}\bar{\theta}_{\bar{m}}{}^\mu\bar{\nabla}_\mu\psi-\bar{\theta}_{\bar{m}}{}^\mu\bar{\nabla}_\mu\bar{\psi}\bar{\gamma}^{\bar{m}}\psi)-m\bar{\psi}\psi\} \tag{34}$$

and the resulting field equations are

$$i\bar{\gamma}^{\bar{m}}\bar{\theta}_{\bar{m}}{}^\mu\bar{\nabla}_\mu\psi+\tfrac{1}{2}i\bar{\gamma}^{\bar{l}}\bar{\theta}_{\bar{l}}{}^\lambda\Theta_\lambda{}^\rho{}_\rho\psi-m\psi=0 \tag{35}$$

$$i\bar{\theta}_{\bar{m}}{}^\mu\bar{\nabla}_\mu\bar{\psi}\bar{\gamma}^{\bar{m}}+\tfrac{1}{2}i\bar{\psi}\bar{\gamma}^{\bar{l}}\bar{\theta}_{\bar{l}}{}^\lambda\Theta_\lambda{}^\rho{}_\rho+m\bar{\psi}=0 \tag{36}$$

The $O(n)$ current is obtained varying S with respect to $\bar{\omega}$:

$$J^{\lambda\bar{m}\bar{n}}=\frac{1}{\sqrt{g}}\frac{\delta S}{\delta\bar{\omega}_{\lambda\bar{m}\bar{n}}}=\frac{i}{4}\,\bar{\theta}_{\bar{l}}{}^\lambda\bar{\psi}\{\gamma^{\bar{l}},\sigma^{\bar{m}\bar{n}}\}\psi \tag{37}$$

The (nonsymmetric) energy-momentum tensor is obtained by varying S with respect to $\bar{\theta}$:

$$T_{\bar{m}}{}^{\bar{n}} = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta \bar{\theta}^{\bar{m}}{}_{\mu}} \theta^{\bar{n}}{}_{\mu}$$

$$= -\frac{i}{2} \bar{\psi}\bar{\gamma}^{\bar{n}}\bar{\theta}_{\bar{m}}{}^{\mu}\bar{\nabla}_{\mu}\psi + \frac{i}{2} \bar{\theta}_{\bar{m}}{}^{\mu}\bar{\nabla}_{\mu}\bar{\psi}\bar{\gamma}^{\bar{n}}\psi$$

$$+\frac{i}{2} \delta_{\bar{m}}{}^{\bar{n}}[\bar{\psi}\bar{\gamma}^{\bar{r}}\bar{\theta}_{\bar{r}}{}^{\rho}\bar{\nabla}_{\rho}\psi - \bar{\theta}_{\bar{r}}{}^{\rho}\bar{\nabla}_{\rho}\bar{\psi}\cdot\bar{\gamma}^{\bar{r}}\psi - m\bar{\psi}\psi] \tag{38}$$

Notice that the last term vanishes when the equations of motion are satisfied. Using the equations of motion and some Dirac algebra, one can find that the current and the energy-momentum tensor satisfy the following conservation laws:

$$(D\otimes\bar{\nabla})_{\lambda}J^{\lambda\bar{m}\bar{n}} - \Theta_{\rho}{}^{\rho}{}_{\lambda}J^{\lambda\bar{m}\bar{n}} + T^{[\bar{m}\bar{n}]} = 0 \tag{39}$$

$$D_{\mu}T_{\lambda}{}^{\mu} - \Theta_{\mu}{}^{\mu}{}_{\rho}T_{\lambda}{}^{\rho} + \Theta_{\rho}{}^{\mu}{}_{\lambda}T_{\mu}{}^{\rho} - \bar{R}_{\bar{m}\bar{n}\lambda\sigma}J^{\sigma\bar{m}\bar{n}} = 0 \tag{40}$$

Here $D\otimes\bar{\nabla}$ denotes the covariant derivative acting on both sets of indices of J, and $\bar{R}$ is the curvature of $\bar{\omega}$. Equation (39) expresses the conservation of angular momentum and equation (40) the conservation of energy-momentum (Hehl *et al.*, 1976).

In this brief review of the gravitational coupling of Dirac spinors, we have described the gravitational field by means of the variables $\bar{\omega}$, $\bar{\theta}$. What would change if we used the variables ω, θ, τ? The action would be

$$\hat{S}(\psi, \bar{\psi}; \omega, \theta, \tau) = S(\psi, \bar{\psi}; \bar{\omega}, \bar{\theta}) \tag{41}$$

with $\bar{\omega}$ and $\bar{\theta}$ given by (4) and (7). Defining the spacetime-dependent Dirac matrices $\gamma^m = \tau_{\bar{r}}{}^m\bar{\gamma}^{\bar{r}}$, which satisfy $\{\gamma^m, \gamma^n\} = 2k^{mn}$, and the spacetime-dependent generators $\sigma^{mn} = \frac{1}{4}[\gamma^m, \gamma^n] = \tau_{\bar{r}}{}^m\tau_{\bar{s}}{}^n\bar{\sigma}^{\bar{r}\bar{s}}$, we would derive the Dirac equation

$$i\gamma^m\theta_m{}^{\mu}[\partial_{\mu}\psi + \tfrac{1}{2}(\omega_{\mu mn} - \tau_{\bar{r}m}\partial_{\mu}\tau^{\bar{r}}{}_n)\sigma^{mn} + \tfrac{1}{2}\Theta_{\mu}{}^{\rho}{}_{\rho}\psi] - m\psi = 0$$

which is obviously the same as equation (35). The current that one obtains by varying $\hat{S}$ with respect to ω is the same as the current (37) except that the barred indices are transformed to unbarred ones by means of τ. The two "energy-momentum tensors"

$$\overset{\theta}{T}_m{}^{\mu} = \frac{1}{\sqrt{g}} \frac{\delta\hat{S}}{\delta\theta^m{}_{\mu}}, \qquad \overset{\tau}{T}_{\bar{m}}{}^r = \frac{1}{\sqrt{g}} \frac{\delta\hat{S}}{\delta\tau^{\bar{m}}{}_r}$$

are easily seen to be related among themselves and to the energy-momentum tensor of equation (38) by

$$\overset{\theta}{T}_m{}^\mu = \theta_n{}^\mu T_m{}^n, \qquad \overset{\tau}{T}_{\bar{m}}{}^r = \tau_{\bar{m}}{}^n[T_n{}^r + (D\otimes\nabla)_\lambda J^\lambda{}_n{}^r - \Theta_\rho{}^\rho{}_\lambda J^\lambda{}_n{}^r]$$

This is a consequence of the larger gauge group of this formulation.

Now let $\bar{\gamma} = \prod_{\bar{r}=1}^n \bar{\gamma}^{\bar{r}}$ and $P_\pm = \frac{1}{2}(1 \pm \bar{\gamma}) : \eta \to \eta_\pm$ be the chiral projectors. When $m = 0$ we define the Weyl operator $\not{D}_{(\bar{\omega},\bar{\theta})} : \Gamma(\eta_+) \to \Gamma(\eta_-)$ by

$$\not{D}_{(\bar{\omega},\bar{\theta})} = (i\bar{\gamma}^{\bar{m}}\bar{\theta}_{\bar{m}}{}^\mu\bar{\nabla}_\mu + \tfrac{1}{2}\bar{\gamma}^{\bar{l}}\bar{\theta}_{\bar{l}}{}^\lambda\Theta_\lambda{}^\rho{}_\rho)P_+ \tag{42}$$

and

$$\hat{\not{D}}_{(\omega,\theta,r)} = \not{D}_{(\bar{\omega},\bar{\theta})} \tag{43}$$

Denote ω^Λ, θ^Λ, τ^Λ the fields transformed as in equations (13)–(15) with $\bar{\Lambda} = 0$, $w = 0$. If we observe that equations (7) and (4) can be written formally $\bar{\omega} = \omega^{\tau^{-1}}$ and $\bar{\theta} = \theta^{\tau^{-1}}$, then the analogy of this definition to equations (24) and (28) becomes immediately clear. When $n = 4k - 2$, the effective action $\hat{\Gamma}(\omega, \theta, \tau) = -\ln\det \hat{\not{D}}_{(\omega,\theta,\tau)}$ is not gauge invariant (Alvarez-Gaumé and Witten (1983)); we regard the anomaly as a one-form on Aut $\xi \oplus \bar{\xi}$. From the very definition [equation (43)] it is clear that the operator $\hat{\not{D}}_{(\omega,\theta,\tau)}$ is invariant under the normal subgroup $\alpha\ \mathrm{Aut}_M^{GL(n)}\xi$; so the same will be true of the effective action

$$\hat{\Gamma}(\omega^\Lambda, \theta^\Lambda, \tau^\Lambda) = \hat{\Gamma}(\omega, \theta, \tau) \tag{44}$$

We define the function Γ on $\mathscr{C}(\bar{\xi}) \times \mathrm{Iso}(TM, \bar{\xi})$ by $\Gamma(\bar{\omega}, \bar{\theta}) = -\ln\det \not{D}_{(\bar{\omega},\bar{\theta})}$. Then $\hat{\Gamma}(\omega, \theta, \tau) = \Gamma(\bar{\omega}, \bar{\theta})$. Let τ_0 and θ_0 be defined by $\tau_0{}^{\bar{m}}{}_r = \delta^{\bar{m}}{}_r$ and $\theta_0{}^m{}_\mu = \delta_\mu{}^m$ in all charts; we shall sometimes write $\tau_0 = 1$, $\theta_0 = 1$. Putting $\Lambda = \tau^{-1}$ or $\Lambda = \theta$ in equation (44), we get

$$\hat{\Gamma}(\omega, \theta, \tau) = \hat{\Gamma}(\bar{\omega}, \bar{\theta}, 1) = \hat{\Gamma}(\Gamma, 1, \bar{\theta}) = \Gamma(\bar{\omega}, \bar{\theta}) \tag{45}$$

with $\Gamma = \omega^\theta$ [see equation (6)]. So the subgroup $\alpha\ \mathrm{Aut}_M^{GL(n)}\xi$ will be anomaly-free; in the "unitary" gauge $\tau = \tau_0$ the anomaly will be in $\gamma^{\tau_0}\,\mathrm{Aut}^{O(n)}\bar{\xi}$.

Explicit calculation in two dimensions shows that the anomaly "originally" lies in the vertical subgroup $\gamma^{\tau_0}\,\mathrm{Aut}_M^{O(n)}\bar{\xi}$ (Langouche, 1984; Leutwyler, 1984). The same is true in higher dimensions (Leutwyler and Mallik, 1986). One expects this to happen on the basis of the analogy with the anomalies of gauge theories (Langouche *et al.*, 1984); furthermore, because of the remark in Section 2, it would generally not make sense to say that the anomaly is in Diff M. This is seen even more clearly if we look at the physical meaning of the gravitational anomalies. The functional derivatives of $\hat{\Gamma}$ (or Γ) with respect to the background fields are the vacuum expectation values of the currents; in analogy to equations (37) and (38) and what followed we define

$$\langle \overset{\theta}{T}_m{}^{\mu}\rangle = \frac{1}{\sqrt{g}}\frac{\delta\hat{\Gamma}}{\delta\theta^m{}_\mu}; \qquad \langle \overset{\tau}{T}_{\bar{m}}{}^{n}\rangle = \frac{1}{\sqrt{g}}\frac{\delta\hat{\Gamma}}{\delta\tau^{\bar{m}}{}_n}; \qquad \langle J^\lambda{}_m{}^n\rangle = \frac{1}{\sqrt{g}}\frac{\delta\hat{\Gamma}}{\delta\omega_\lambda{}^m{}_n}$$
$$\langle T_{\bar{m}}{}^{\mu}\rangle = \frac{1}{\sqrt{g}}\frac{\delta\Gamma}{\delta\bar{\theta}^{\bar{m}}{}_\mu}; \qquad \langle \bar{J}^\lambda{}_{\bar{m}}{}^{\bar{n}}\rangle = \frac{1}{\sqrt{g}}\frac{\delta\Gamma}{\delta\bar{\omega}_\lambda{}^{\bar{m}}{}_{\bar{n}}} \tag{46}$$

Only two of these quantities are independent: using equation (4) and (7), one finds that

$$\langle J^\lambda{}_m{}^n\rangle = \tau^{\bar{r}}{}_m\tau_{\bar{s}}{}^n\langle \bar{J}^\lambda{}_{\bar{r}}{}^{\bar{s}}\rangle$$
$$\langle \overset{\theta}{T}_m{}^{\mu}\rangle = \tau^{\bar{r}}{}_m\langle T_{\bar{r}}{}^{\mu}\rangle$$
$$\langle \overset{\tau}{T}_r{}^{n}\rangle = \langle T_r{}^n\rangle + (D\otimes\nabla)_\lambda\langle J^\lambda{}_r{}^n\rangle - \Theta_\rho{}^\rho{}_\lambda\langle J^\lambda{}_r{}^m\rangle$$

From now on we take J to be the current and T the energy-momentum tensor.

If we now vary $\hat{\Gamma}$ with respect to an arbitrary infinitesimal transformation of $\mathrm{Aut}^{GL(n)\times O(n)}\xi\oplus\bar{\xi}$ and use these relations we find

$$\begin{aligned}\delta(\varepsilon,\bar{\varepsilon},w)\hat{\Gamma} = \int d^4x\,\sqrt{g}\{&-\bar{\varepsilon}_{\bar{m}\bar{n}}[\langle T^{\bar{m}\bar{n}}\rangle + (D\otimes\bar{\nabla})_\lambda\langle J^{\lambda\bar{m}\bar{n}}\rangle - \Theta_\rho{}^\rho{}_\lambda\langle J^{\lambda\bar{m}\bar{n}}\rangle]\\ &- w^\nu\bar{\omega}_{\nu\bar{m}\bar{n}}[\langle T^{\bar{m}\bar{n}}\rangle + (D\otimes\bar{\nabla})_\lambda\langle J^{\lambda\bar{m}\bar{n}}\rangle - \Theta_\rho{}^\rho{}_\lambda\langle J^{\lambda\bar{m}\bar{n}}\rangle]\\ &- w^\nu[D_\mu\langle T_\nu{}^\mu\rangle + \Theta_\mu{}^\rho{}_\nu\langle T_\rho{}^\mu\rangle - \Theta_\mu{}^\mu{}_\rho\langle T_\nu{}^\rho\rangle - R^\rho{}_{\sigma\nu\lambda}\langle J^\lambda{}_\rho{}^\sigma\rangle]\}\end{aligned} \tag{47}$$

We observe that ε drops off in accordance with the previous remark. Comparison with equation (39) and (40) now shows the following: an anomaly for the subgroup $\mathrm{Aut}_M^{O(n)}\bar{\xi}$ implies nonconservation of angular momentum and an anomaly for Diff M (whenever this subgroup is defined) implies nonconservation both of energy-momentum and angular momentum. This is again related to the remarks in section 2, for a "purely Diff M" anomaly in one trivialization would contain an $\mathrm{Aut}_M^{O(n)}\bar{\xi}$ part in another trivialization.

We now make contact with the work in Bardeen and Zumino (1984). Assume that M is parallelizable and fixed global fields of frames are given. For each $\kappa\in\mathrm{Riem}\,\xi$, there is a unique transformation $\bar{\Lambda}$ in the subgroup $\gamma\,\mathrm{Aut}_M^{O(n)}\bar{\xi}$ which makes the matrix $\tau^{\bar{m}}{}_r$ symmetric. The residual gauge group is (isomorphic to) $\mathrm{Aut}^{GL(n)}\xi$; it consists of transformations (Λ, f) belonging to the subgroup $\tilde{\alpha}\,\mathrm{Aut}^{GL(n)}\xi$ defined in Section 2, followed by a field-dependent transformation $\bar{\Lambda}=\bar{\Lambda}(\Lambda,\tau)$ in $\gamma\,\mathrm{Aut}_M^{O(n)}\bar{\xi}$ satisfying $\Lambda^T\tau\bar{\Lambda} = \bar{\Lambda}^{-1}\tau\Lambda$, which restores the symmetry of τ. In the symmetric gauge, τ is uniquely determined from κ; therefore, we now think of $\hat{\Gamma}$ as a functional of ω, θ, and κ.

Although isomorphic to the anomaly-free normal subgroup $\alpha\,\mathrm{Aut}_M^{GL(n)}\xi$, this subgroup $\mathrm{Aut}_M^{GL(n)}\xi$ is really different because it implies a readjustment of the $\mathrm{Aut}_M^{O(n)}\bar{\xi}$ gauge. As a consequence, $\mathrm{Aut}_M^{GL(n)}\xi$ is now anomalous. To see this more clearly, let us first fix part of the residual gauge invariance by going to the "n-bein gauge" $\kappa=\kappa_0=1$ (see Section 1). The residual gauge freedom is now the group $\mathrm{Aut}^{O(n)}\xi$ given by transformations of the form (Λ, Λ, f) with $\Lambda: M \to O(n)$. The symmetric n-bein for κ_0 is $\tau_0=1$, so the group $\mathrm{Aut}^{O(n)}\xi$ has the same anomaly as $\gamma^{\tau_0}\,\mathrm{Aut}^{O(n)}\bar{\xi}$:

$$\begin{aligned}\hat{\Gamma}(\omega^\Lambda, \theta^\Lambda, \kappa_0^\Lambda) &= \hat{\Gamma}(\omega^\Lambda, \theta^\Lambda, \kappa_0) = \Gamma(\omega^\Lambda, \theta^\Lambda) \\ &= \Gamma(\omega, \theta) + \int_1^\Lambda \alpha^{O(n)} = \hat{\Gamma}(\omega, \theta, \kappa_0) + \int_1^\Lambda \alpha^{O(n)} \end{aligned} \tag{48}$$

The induced anomaly in $\mathrm{Aut}_M^{GL(n)}\xi$ is an extension of the induced anomaly in $\mathrm{Aut}_M^{O(n)}\xi$. The one-form $\alpha^{O(n)}$ on $\mathrm{Aut}^{O(n)}\xi$ naturally extends to a one-form $\alpha^{GL(n)}$ on $\mathrm{Aut}^{GL(n)}\xi$; at the level of secondary characteristic classes, one continues the invariant polynomials on $\mathrm{Lie}\,O(n)$ to invariant polynomials on $\mathrm{Lie}\,GL(n)$ (Langouche *et al.*, 1984). Then, one has instead of equation (44)

$$\hat{\Gamma}(\omega^\Lambda, \theta^\Lambda, \kappa^\Lambda) = \hat{\Gamma}(\omega, \theta, \kappa) + \int_1^\Lambda \alpha^{GL(n)} \tag{49}$$

One can also fix part of the $\mathrm{Aut}^{GL(n)}\,\xi$-gauge by choosing the "metric gauge" $\theta=\theta_0=1$ (see Section 1). The stabilizer of θ_0 is a subgroup of $\mathrm{Aut}^{GL(n)}\xi$ isomorphic to Diff M; it acts on the fields as in equations (13)-(15) with $\Lambda=\partial x/\partial x'$ and $\bar{\Lambda}=\bar{\Lambda}(\Lambda, \tau)$. As a consequence, there is an induced anomaly in Diff M (Langouche *et al.*, 1984). The metric gauge and the n-bein gauge are related by a transformation of $\mathrm{Aut}^{GL(n)}\xi$ with $\Lambda=\bar{\theta}^{-1}$. From equation (49) we obtain, instead of equation (45)

$$\hat{\Gamma}(\bar{\omega}, \bar{\theta}, 1) = \hat{\Gamma}(\Gamma, 1, g) + \int_1^{\bar{\theta}^{-1}} \alpha^{GL(n)} \tag{50}$$

On the lhs one has the effective action in the n-bein gauge, which has an anomalous variation under $\mathrm{Aut}^{O(n)}\xi$; on the rhs the effective action in the metric gauge (a functional of the connection $\Gamma_\lambda{}^\mu{}_\nu$ in TM and the Riemannian structure $g_{\mu\nu}$), which has an anomalous variation under Diff M. They are related by the addition of a local functional of ω, θ, and κ, a gravitational analog of the Wess-Zumino-Witten action.

ACKNOWLEDGMENTS

I am grateful to R. Jackiw and S. Rajeev for several helpful conversations. This work was supported by a fellowship from the Istituto Nazionale di Fisica Nucleare and by the U.S. Department of Energy under contract DEAC02-76ER03069.

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